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Perturbative derivation of exact superpotential for meson fields from matrix theories with one flavour

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Abstract: We derive the superpotential of gauge theories having matter fields in the fundamental representation of gauge fields by using the method of Dijkgraaf and Vafa. We treat the theories with one flavour and reproduce a well-known non-perturbative superpotential for meson field.

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Recently, Dijkgraaf and Vafa [1] proposed a general prescription for computing superpotentials via planar diagrams of matrix theories. Several tests have been achieved for this conjecture [2]–[10] mainly for theories with potentials for the fields in the adjoint representation of the gauge group. A recent paper [11] proved a correspondence for matter fields in the adjoint representation. It is interesting to obtain effective superpotentials for fields with fundamental representations since the non-perturbative superpotentials for these fields are well known (See, for example, Ref. [12]). A typical example is the theories for $N_f < N_c$. In this case, a unique superpotential is known as [13, 14, 15, 12]

$$W_{NP} = (N_c - N_f)\left(\frac{\Lambda^{3N_c-N_f}}{\det \tilde{Q}Q}\right)^{1/(N_c-N_f)}, \tag{1}$$

where $\tilde{Q}, Q$ fields are the fields in the fundamental representation of gauge fields.

Recently, R. Argurio, V.Campos, G.Ferretti and R.Heise [16] evaluated the superpotential for $N_c = 2$ and $N_f = 1$ and compare it to the field theoretical results. They found the agreement up to sixth-order of the coupling constant. It is striking that we can reproduce the well-known superpotentials from the matrix technique.

In this short paper, we will show that matrix results reproduce the non-perturbative superpotentials (1) as well as tree level potentioal for $N_f = 1$. We will mainly follow the notation of ref. [16].

A prescription by Dijkgraaf and Vafa [1] is the followings. For computing the effective superpotential for the glueball field $S = -\frac{1}{32\pi^2}\mathrm{tr}W^\alpha W_\alpha$ in the case of a $\mathrm{U}(N_c)$ gauge group and chiral fields Φ_i in the adjoint representation interacting with a tree level superpotential $W_{\mathrm{tree}}(\Phi_i, \lambda_a)$, we need to compute the matrix integral to leading order in N_c:

$$e^{-\frac{N_c^2}{S^2}\mathcal{F}_{\chi=2}(S,\lambda_a)} \approx \int d\Phi_i\, e^{-\frac{N_c}{S}W_{\mathrm{tree}}(\Phi_i,\lambda_a)}, \tag{2}$$

where we have denoted by λ_a the coupling constants appearing in the superpotential. The effective superpotential is, in this case [1]:

$$W_{DV}(S,\Lambda,\lambda_a) = N_c S\left(-\log\left(\frac{S}{\Lambda^3}\right)+1\right) + N_c\frac{\partial\mathcal{F}_{\chi=2}(S,\lambda_a)}{\partial S}, \tag{3}$$

where the presence on $N_c\frac{\partial}{\partial S}$ is justified by the combinatorics of diagrams written on surfaces with spherical topology. The first piece of the superpotential is the Veneziano-Yankielowicz superpotential for pure $\mathrm{SU}(N_c)$ SYM [17], while the second piece which starts with $O(S^2)$ terms gives the instantonic corrections. In the case that the matrix model is integrable we can write the exact effective superpotential in closed form, otherwise, we can compute it at any given order in S. Recent checks and developments of the conjecture have been performed in [2]–[10].

It is natural to extend the conjecture to theories including one matter chiral multiplets in the fundamental representation of $\mathrm{U}(N_c)$. This is implemented simply by including surfaces with boundaries. To be specific, in the case of gauge group $\mathrm{U}(N_c)$, for adjoint matter Φ_i and fundamental matter Q and $\tilde{Q}$ one should first compute

$$e^{-\frac{N_c^2}{S^2}\mathcal{F}_{\chi=2}(S,\lambda_a)-\frac{N_c}{S}\mathcal{F}_{\chi=1}(S,\lambda_a)} \approx \int d\Phi_i\, dQ\, d\tilde{Q}\, e^{-\frac{N_c}{S}W_{\mathrm{tree}}(\Phi_i,Q,\tilde{Q},\lambda_a)}, \tag{4}$$

and then write (the non-orientable contribution $G_{\chi=1}(S,\lambda_a)$ is absent in this case):

$$W_{DV}(S,\Lambda,\lambda_a) = N_c S\left(-\log\left(\frac{S}{\Lambda^3}\right)+1\right) + N_c\frac{\partial\mathcal{F}_{\chi=2}(S,\lambda_a)}{\partial S} + \mathcal{F}_{\chi=1}(S,\lambda_a)\,. \tag{5}$$

We can easily evaluate the superpotential. We take a $\mathrm{U}(N_c)$ gauge theory with one adjoint chiral multiplet Φ and one chiral multiplets in the fundamental Q and $\tilde{Q}$. The tree level superpotential gives masses to all matter fields, and moreover there is a cubic coupling between the fundamentals and the adjoint. Other possible couplings are turned off. The tree level superpotential is

$$W_{\text{tree}} = \frac{1}{2}M\,\mathrm{tr}\Phi^2 + mQ\tilde{Q} + gQ\Phi\tilde{Q}\,, \tag{6}$$

where the color indices are not written explicitly.

Since there are no self interactions of the adjoint field Φ, all diagrams will involve at least one flavor loop that is a boundary. As a consequence, the genus zero piece of the matrix integral vanishes, $\mathcal{F}_{\chi=2}=0$. To leading order in N_c, the matrix integral is thus saturated by planar diagrams with one boundary, which sum up to $\mathcal{F}_{\chi=1}$.

The matrix integral can thus be easily performed. We can write

$$Z = \int d\Phi_i\,dQ\,d\tilde{Q}\;e^{-\frac{N_c}{S}\left(\frac{1}{2}M\mathrm{tr}\Phi^2+mQ\tilde{Q}+gQ\Phi\tilde{Q}\right)} = \left\langle e^{-\frac{N_c}{S}gQ\Phi\tilde{Q}}\right\rangle, \tag{7}$$

where the correlators are normalized in the form:

$$\langle Q_\alpha\tilde{Q}^\beta\rangle = \frac{1}{m}\frac{S}{N_c}\delta_\alpha^\beta\,, \qquad \langle\Phi_\beta^\alpha\Phi_\lambda^\gamma\rangle = \frac{1}{M}\frac{S}{N_c}\delta_\lambda^\alpha\delta_\beta^\gamma\,. \tag{8}$$

Expanding the exponential, we get

$$\left\langle e^{-\frac{N_c}{S}gQ\Phi\tilde{Q}}\right\rangle = \sum_{k=0}^{\infty}\frac{1}{(2k)!}\left(\frac{gN_c}{S}\right)^{2k}\left\langle(Q\Phi\tilde{Q})_1(Q\Phi\tilde{Q})_2\ldots(Q\Phi\tilde{Q})_{2k}\right\rangle, \tag{9}$$

where we took into account that only correlators of an even number of fields Φ are non zero.

It is now a simple combinatorial problem to extract from (9) the coefficients of the connected planar diagrams with one boundary. The different diagrams can be obtained first by contracting the Qs and $\tilde{Q}$s in $(2k-1)!$ ways to give a single boundary, and then connecting $2k$ points on the boundary through k non intersecting lines (the $\langle\Phi\Phi\rangle$ propagators). The solution to this last combinatorial problem can be found in [18], eq. (31). The result for the free energy is:

$$\frac{N_c}{S}\mathcal{F}_{\chi=1}(S,g,m,M) = -\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}\left(\frac{gN_c}{S}\right)^{2k}\left(\frac{S}{mN_c}\right)^{2k}\left(\frac{S}{MN_c}\right)^{k}N_c^{k+1}\,, \tag{10}$$

which we can rewrite, for $\alpha = \frac{g^2}{m^2M}$, as:

$$\mathcal{F}_{\chi=1}(S,\alpha) = -\sum_{k=1}^{\infty}\frac{(2k-1)!}{(k+1)!k!}\alpha^k S^{k+1}\,. \tag{11}$$

This expression can actually be summed to give:

$$\mathcal{F}_{\chi=1}(S,\alpha) = -S\left[\frac{1}{2} + \frac{1}{4\alpha S}\left(\sqrt{1-4\alpha S}-1\right) - \log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-4\alpha S}\right)\right]. \tag{12}$$

In this way, we can obtain the superpotential of the form:

$$W_{DV} = N_c S\left(-\log\left(\frac{S}{\Lambda^3}\right)+1\right) + \mathcal{F}_{\chi=1}(S,\alpha) \tag{13}$$

which is expected to be the exact superpotential for our theory with one flavor and the Yukawa coupling to the adjoint matter field. This is a result obtained in ref. [16]. The above form of superpotential seems rather complicated. Therefore, some simplification must be required.

We are now going to rewrite to the superpotential for meson fields. The equation of the gluino fields can be obtained from $\partial_S W_{DV} = 0$:

$$\frac{W'}{\Lambda^3} = -\log\left[\frac{\left(S/\Lambda^3\right)^{N_c}}{\left(1+\sqrt{1-4\alpha S}\right)/2}\right] = 0 \tag{14}$$

which implies

$$\left(\frac{S}{\Lambda^3}\right)^{2N_c} - \left(\frac{S}{\Lambda^3}\right)^{N_c} + a\left(\frac{S}{\Lambda^3}\right) = 0, \tag{15}$$

where $a = \alpha\Lambda^3$. Using this relation, We can rewrite the superpotential in the form

$$W_{DF} = (N_c-1)S + \Lambda^{3N_c}S^{-N_c+1} - \frac{g^2}{2m^2M}\Lambda^{6N_c}S^{-2N_c+2}\,. \tag{16}$$

In this form of the potential, we can obtain the equation of motion (15) and the same value as the original form of the potential when all fields are integrated out. In order to recover a matter field, we use a matching relation$\Lambda^{3N_c} = m\tilde{\Lambda}^{3N_c-1}$ and write the superpotential in the form:

$$W_{DF} = (N_c-1)S + m\tilde{\Lambda}^{3N_c}S^{-N_c+1} - \frac{g^2}{2M}\tilde{\Lambda}^{6N_c-2}S^{-2N_c+2}\,. \tag{17}$$

The expectation value of $X = Q\tilde{Q}$ fields can be given by $X = \partial_m W_{DV}$, which implies

$$X = \tilde{\Lambda}^{3N_c-1}S^{-N_c+1}\,. \tag{18}$$

By using this fields, we can write the superpotential as

$$W_{DV} = (N_c-1)\left(\frac{\tilde{\Lambda}^{3N_c-1}}{X}\right)^{1/(N_c-1)} + mX - \frac{g^2}{2M}X^2\,. \tag{19}$$

Now we can easily interpret the origin of each term of the superpotential. Consider the $\mathrm{U}(N_c)$ theory with an adjoint matter field Φ and $N_f = 1$ chiral fields in the fundamental Q and $\tilde{Q}$.

The tree level superpotential is given by

$$W_{\text{tree}} = \frac{1}{2}M\,\mathrm{tr}\Phi^2 + mQ\tilde{Q} + gQ\Phi\tilde{Q}\,. \tag{20}$$

Let us first integrate out the adjoint field Φ. This trivially gives $\Phi = -\frac{g}{M}\tilde{Q}Q$, and substituting into (20) we find,

$$W_{\rm tree} = mX - \frac{1}{2}\frac{g^2}{M}X^2 \,. \tag{21}$$

The first term in (19) represents the exact superpotential obtained in [13, 14]:

$$W_{NP} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det \tilde{Q}Q}\right)^{1/(N_c - N_f)} , \tag{22}$$

which can be written in the case $N_f = 1$ as

$$W_{NP} = (N_c - 1)\left(\frac{\hat{\Lambda}^{3N_c - 1}}{X}\right)^{1/N_c - 1} . \tag{23}$$

Therefore, DV superpotential reproduces the famous superpotential (22) for $N_c = 1$ as well as the tree level potential(20). Namely,

$$W_{DV} = W_{NP} + W_{\rm tree} \,. \tag{24}$$

Note that our transformation can be justified at the level of the superpotential integrated out all the fields. Namely, we have identified the superpotential when the superpotentials are written in the variables Λ, m, g and M. However, we can easily integrate in the original fields by the usual procedure.

In this paper, we have shown that the superpotentials for meson fields can be derived from matrix technique. We must investigate whether "Large N" or just a "planar" for the matrrix-superpotential correspondence.

Lastly, we are going to evaluate superpotentials explicitly. First of all, we choose a variable $y = (\frac{m}{\Lambda^3}X)^{1/(N_c - 1)}$. Then the superpotential can be written as

$$W/\Lambda^3 = (N_c - 1)y^{-1} + y^{N_c - 1} - \frac{1}{2}ay^{2N_c - 2} , \tag{25}$$

The equation of motion can be written in the form

$$f(y) = 1 - y^{N_c - 1} + ay^{2N_c - 1} = 0 \,. \tag{26}$$

As a vacuum solution, we choose a solution around $y = 1$ for small value of a. Note also that $W'(y) = -(N_c - 1)\Lambda^3 f(y)/y^2$ The value of superpotential W at the pole of $f(y)$ can be evaluated as

$$W = \oint \frac{dy}{2\pi i} W(y) \frac{f'(y)}{f(y)} , \tag{27}$$

where the contour encircles a zero of the function $f(y)$. By performing partial integration, we can write

$$W = (N_c - 1)\Lambda^3 \oint \frac{dy}{2\pi i} y^{-2} f(y) \log f(y) \,. \tag{28}$$

We use the integral representation for the logarithm and write

$$W = (N_c - 1)\Lambda^3 \oint \frac{ds}{2\pi i s^2} \oint \frac{dy}{2\pi i} y^{-2} f(y)^{s+1}\,, \tag{29}$$

where the contour of s encircles the point $s = 0$. Since the contour integral of the variable y has a cut by the introduction of the variable s, We can represent it as a line integral between 0 to a pole which shrinks to y=1 upon expanding with respect to a, therefore, we can write

$$W \,=\, (N_c - 1)\Lambda^3 \oint \frac{ds}{2\pi i s^2} \frac{\sin \pi(s+1)}{\pi} \int_0^1 dy y^{-2} \left(1 - y^{N_c} + a y^{2N_c - 1}\right)^{s+1} . \tag{30}$$

By changing variables to $y = t^{1/N_c}$ and expanding the superpotential with respect to a, we can evaluate the t-integration by the usual beta integral. We also find the integral with respect to s has just a single pole, and we finally get an expression of superpotential:

$$W \,=\, N_c \Lambda^3 \sum_{n=0}^{\infty} \frac{\Gamma((2 - 1/N_c)n - 1/N_c)/\Gamma(-1/N_c)}{\Gamma((1 - 1/N_c)n + 2 - 1/N_c)/\Gamma(2 - 1/N_c)n!} a^n \tag{31}$$

$$= N_c \Lambda^3 \left[1 - \frac{1}{2N_c} a - \frac{N_c - 1}{2N_c^2} a^2 - \frac{(5N_c - 4)(N_c - 1)}{6N_c^3} a^3 + \cdots \right], \tag{32}$$

The first few terms perfectly agrees with the result for $N_c = 2$ [16].

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